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TRAJECTORY DESIGN METHODS

Report to

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Missile and Spacecraft Aerodynamics

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JET PROPULSION LABORATORY
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CONTENTS

CONTENTS (Cont'd)

I. INTRODUCTION

This report describes the tools and techniques of existing trajectory design methods of the Jet Propulsion Laboratory. The trajectory calculation techniques presented are those believed to be of particular interest to the Committee on Missile and Spacecraft Aerodynamics.

II. TRAJECTORY PROGRAMS

A. Powered Flight Trajectory Computing Programs

1. General purpose program. An existing general-purpose computing program satisfies the major portion of the Laboratory's requirements for powered-flight trajectory generation. In this program, the missile is considered as a point mass with 3 degrees of freedom and has the capability of handling a maximum of six stages of powered flight with any amount of interstage coasting.

The equations of motion include gravitational, aerodynamic, and thrust forces: The gravitational forces are functions of latitude and earth-centered radial distance because the Earth is considered as an oblate spheroid; aerodynamic forces are computed separately as drag force and normal force. The aerodynamic-force coefficients may be stored either as tables or as cubic polynomials of the Mach number. When the polynomial form is used, provision is made to describe each aerodynamic coefficient by four polynomials for four different ranges of Mach number. The atmosphere is assumed to rotate as a rigid body. The ARDC Standard Atmosphere of 1957 is used for altitudes below 300,000 ft. Above this altitude, the pressure ratio is described by a logarithmic equation and the acoustic velocity is assumed to be constant. The magnitude of the thrust force is computed from the vacuum thrust reduced by the atmospheric back-pressure at altitude. A constant value of propellant consumption rate is generally assumed for each stage.

Several options are available for thrust-direction control in both the pitch and yaw planes. A zero-lift pitch program is available in which the thrust vector is maintained colinear with the velocity vector. Options are available to fly with either angle of attack or inertial attitude described by seventh-order polynomials in time. All of these options are available for control of both pitch and yaw.

The equations of motion are integrated in an inertial Cartesian coordinate system by a fourth-order Runge-Kutta routine. Suitable transformations are available to provide Earth-fixed velocity and position. The primary coordinate system is chosen so as to be compatible with guidance-studies requirements. The origin is located at the launch pad, and the axes point vertically, downrange and crossrange. Transformations are made to both inertial and Earth-fixed earth-centered coordinate systems. The geocentric Earth-fixed coordinate system is a convenient branch point for computing tracking-station referenced coordinates. The last-stage burnout position and velocity in the geocentric inertial coordinate system are used as inputs to one of several available earth-satellite, lunar, or interplanetary coasting trajectory programs. The tie-in between the powered flight program and the coasting programs is completely automatic.

At the present time, the powered flight program has only a 3 degree of freedom capability; however, 6 degree of freedom capability is easily attained by the inclusion of three moment equations.

2. Optimization of attitude during burning. For payload maximization, an available computing program, which computes in a non-rotating spherical Earth-coordinate system without atmosphere, determines, subject to various restrictions on end conditions, the optimum attitude during burning. Position is specified by two coordinates: r, the geocentric radial distance, and θ, the downrange angle. The integration of the equations of motion proceeds from the initial conditions r_0, θ_0, $\dot{r}_0$, and $\dot{\theta}_0$. During the integration, iterative procedures are used to find the attitude which provides a maximization of burnout energy. Options exist for optimizing the attitude subject to various end-condition restrictions, which include specifying a burnout radial distance, a burnout velocity pitch angle, and a specific value of $\Delta\theta$ during burning. These restrictions may be applied either singly or in any combination; however, as the number of restrictions applied increases, the speed of computation decreases.

3. Azimuth rotation program. A program exists which satisfies the frequent need to rotate an existing trajectory to a different firing azimuth. The existing trajectory is transformed point by point to a coordinate system which is natural to the launch-pad location and firing azimuth; i.e., one axis of the coordinate system lies along the firing azimuth. The effect of Earth rotation on both position and velocity is removed at each point of the trajectory, which gives as an intermediate result, a non-rotating Earth trajectory. The reverse process is then applied to add-in initial inertial velocity on the new firing azimuth. An option is available not only to rotate the firing azimuth, but to move the location of the launch site as well. It is also possible to enter this rotation program at an intermediate point with a non-rotating Earth trajectory in order to produce a rotating

Earth trajectory on any desired firing azimuth from any chosen launch location. This program is suitable for use with trajectories where the total downrange arc length is no greater than about 15 deg.

B. Conic Interplanetary Coasting Program

1. Patched conic program. An available interplanetary-coasting trajectory program, which calculates interplanetary trajectories by a patched-conic method, is three dimensional and generates planetary positions from stored orbital elements. As the computation is performed in a forward going (in time) conventional manner, it is particularly adaptable to being attached to the output of a powered flight-trajectory computing program.

The computation begins with the specification of seven Earth-fixed input parameters; injection time, three position coordinates, and three velocity components, which are used to generate a geocentric hyperbola out to a geocentric distance of approximately one million miles. At this point, the geocentric hyperbolic trajectory is converted to a heliocentric ellipse.

A homing procedure, which is an integral part of this program, is guided by twenty-one partial derivatives of planet-miss-distance components with respect to the seven input parameters. The standard homing procedure allows the seven input parameters to so be adjusted as to achieve a near-miss at the target planet. As the partial derivatives or miss coefficients may be computed at any point on the trajectory, the program is a useful tool for investigating mid-course guidance corrections.

Basically, this program is a conic simulation of the precise integration procedures which are used to generate interplanetary trajectories. The program accepts the same type of input information as the integration program and uses the same type of homing procedures as are used with the integration program. The patched conic program obviously does not have the accuracy of the integration program, but its clear advantage in computing speed makes it useful for studying and roughing-in trajectories.

2. Heliocentric transfer ellipse program. A second existing interplanetary-coasting trajectory program takes a rather unconventional approach to the problem. This program is three dimensional and uses a published planetary ephemeris.

The program is set into operation by the specification of target planet, desired arrival date, and desired heliocentric transfer central angle as inputs, and the ephemeris is consulted for target-planet position and Earth position at the arrival date. These positions are differenced to produce the communications distance, which is an important trajectory design parameter. The heliocentric central angle, combined with target-planet position at arrival time, locates the position of Earth at injection. The ephemeris is entered with this Earth position in order to find the time at which Earth is at this position; this determines a heliocentric ellipse. The various parameters of this ellipse are determined, particularly the required velocity at the heliocentric injection point. The difference between this required velocity and the Earth velocity is the incremental velocity which is needed at heliocentric injection. The three components of heliocentric injection velocity and the heliocnetric injection time specify the coupling conditions that must be met by the geocentric coasting trajectory and the powered flight.

Any trajectory in this program is a "hit" trajectory; i.e., the need for homing procedures is eliminated, which is a major advantage. It is another major advantage that only four injection conditions need be matched by the geocentric trajectory as opposed to the seven which must be matched in the forward-going interplanetary program.

An option exists for automatically scanning the input-parameter heliocentric transfer central angle to find the transfer ellipse which requires minimum incremental energy for any given arrival date. This option allows tables and plots to be constructed which show all the significant trajectory parameters as a function of arrival date (communication distance) for the minimum energy transfer which arrives at that date.

This program has been used to investigate heliocentric elliptical transfer trajectories to Mars, Venus, and Mercury for the period 1960 - 1965. The arrival-time spans following the Mars 1960 opposition and the Venus 1961 conjunction have been investigated fully. For each arrival date, transfer trajectories have been generated at 5-deg intervals of heliocentric central angles throughout the entire elliptical range. Every significant trajectory parameter is listed for each case. This investigation has been extended to cover all the synodic periods of Mars, Venus, and Mercury during 1961 - 1965; however, this extended survey examines only the minimum energy trajectory for each arrival date. Arrival dates were examined at one-day intervals starting at the date of inferior conjunction for the interior planets and opposition for the exterior planets. The arrival-date span investigated is about 6 months for Mars and Venus and about 1 month for Mercury.

Graphical presentations of the minimum energy trajectories will be generally available within a short time.

This program is to be part of a new automatic interplanetary-trajectory design program which will treat the powered flight portion of the flight, the geocentric coasting hyperbola, and the heliocentric coasting ellipse. The Mars and Venus trajectories described previously have been stored on magnetic tape. From this tape, a trajectory is selected by arrival date and heliocentric central angle with the choice being guided by the aforementioned minimum-energy trajectory-parameter plots. The tape lists the 4 heliocentric transfer-point matching parameters for each trajectory. Because the trajectory selection parameters have been chosen at such close intervals, simple interpolation routines may be used to select any trajectory. In effect, all reasonable heliocentric trajectories to Mars and Venus during 1960 - 1965 have been precomputed, catalogued, and are available in a form which permits their ready use.

C. Methods of Integration

For integrating the equations of motion, the JPL trajectory program uses a geocentric, space-fixed (i. e. space-orientation fixed) rectangular coordinate system, referred to the mean equator and equinox of 1950.0. The gravitational effects of the Moon, the Sun, and as many planets as are appropriate are brought in by using their ephemerides referred to the mean equator and equinox of 1950.0. Two oblateness terms for the Earth are included in the equations of motion. The numerical integration is accomplished by a combination of a Runge-Kutta method (used for starting and changing interval size) and a Gauss-Jackson (second sum) method of the open formula type, with differences up to the seventh order retained (correct through h^{10}, where h is the integration step size). The input is normally in Earth-fixed spherical coordinates (radial distance, latitude, longitude, velocity magnitude, velocity azimuth, and velocity elevation angle), although space-fixed rectangular coordinates can also be used. Outputs are available in the same two types of coordinate systems. Predicted values of observables include corrections for station anomalies, for atmosphere, and for time interval of transmission.

The Laboratory also has available the Themis program and a modified S.T.L. program. The modified S.T.L. program is divided into three phases, each using space-fixed equatorial rectangular coordinates, but with different origins: geocentric, heliocentric, and planetocentric (Mars or Venus). Transfer from one phase to another occurs at pre-assigned distances from the Earth and from the target planet. During the geocentric phase, only the disturbing effects of the Sun, the Moon and the oblateness of the Earth are included. During the heliocentric phase, the Earth-Moon system, Jupiter, Venus, and Mars are the disturbing bodies. During the planetocentric phase, Jupiter and the Sun are the disturbing bodies. In the geocentric phase, the ephemeris of the Sun is referred to the mean equator and equinox of 1959.0, and the ephemeris of the Moon is referred to the equator and equinox of date. In the other phases, all ephemerides are referred to the equator and equinox of 1950.0. In the geocentric phase, input and output may be in space-fixed spherical or rectangular coordinates, or in Earth-fixed spherical coordinates. In the other two phases, input and output are in heliocentric rectangular or planetocentric rectangular coordinates, referred to the equator and equinox of 1950.0. The numerical integration procedure uses

Runge-Kutta for starting and changing interval size, and a Milne predictor corrector difference scheme, which is useful in second-order equations not explicitly containing the first derivative; the Milne scheme is correct through h^5.

The Themis program is a double precision program in space-fixed rectangular coordinates. It may be used in either geocentric or heliocentric form, but there is no automatic transfer from one phase to the other. When used in geocentric form, only the Sun and the Moon are disturbing bodies. When used heliocentric, the Earth-Moon system and all other planets except Mercury can be used as perturbing bodies. Input and output are restricted to rectangular coordinates. The integration procedure is a Gauss-Jackson (second sum) method, retaining differences up to the tenth order, with a correction formula which may be applied as many as seven times. Starting values are obtained in this case by generating an approximate solution initially from the osculating elements at epoch and iterating on these with the correction formula to obtain accurate starting values.

For lunar satellites, there is being planned an additional system of equations obtained by translating the rectangular coordinate system of 1950.0 to the center of the Moon and providing for a time-varying potential function for the Moon, treated as a rotating ellipsoid. Transformations are to be provided to obtain selenographic latitudes and longitudes relative to the mean central point of the visible disc and the mean equator of the Moon.

D. Astronomical Constants

The table of constants presented here is intended primarily for use in JPL trajectory programs. The column headed "Recommended constant," is a list of what are considered to be the best available set of constants for use in the trajectory program if they were to be respecified. No recommendation is made that all present values be changed to these values, because in all cases, the differences result in errors which are sufficiently small for present JPL requirements for pre-flight trajectories. In-flight analysis of tracking data for missions utilizing mid-course maneuvers will impose accuracy requirements so severe that knowledge of a progressively larger number of physical constants must be improved. (See Table 1.)

Table 1. Astronomical Constants

Constant	Now Used in JPL Trajectory Program	Recommended Constants	Ref.
GM_S	$1.32534215\ 10^{20}\ \frac{m^3}{sec^2}$	$1.3253 \times 10^{20}\ \frac{m^3}{sec^2}$	
		(Using $R_0 = 6378.260$ km and $\pi_\odot^* = 8.7984''$)	(1) (2)
GM_E	$3.9861353 \times 10^{14}\ \frac{m^3}{sec^2}$	$3.986135 \times 10^{14}\ \frac{m^3}{sec^2}$	(3)
mass $\frac{Earth}{Sun}$	3.007627×10^{-6}	3.00772×10^{-6}	
GM_{moon}	$4.898477\ 10^{12}\ \frac{m^3}{sec^2}$	$4.89820 \times 10^{12}\ \frac{m^3}{sec^2}$	
mass $\frac{moon}{earth}$	$0.012288787 = \frac{1}{81.375}$	$\frac{1}{81.378}$	(4)

* $\pi_\odot$ is the solar parallax. $\pi_\odot = \frac{R_0}{1\ A.U.\ \sin 1''}$

Table 1 (Cont'd)

Constant	Now Used in JPL Trajectory Program	Recommended Constants	Ref.
GM_{Mars}	$0.430207 \times 10^{10} \frac{m^3}{sec^2}$	$0.4272 \times 10^{14} \frac{m^3}{sec^2}$	
mass $\frac{Mars}{Sun}$	$\frac{1}{3,078,818}$	$\frac{1}{3,102,000}$	(5)
GM_{Venus}	$0.324124 \times 10^{15} \frac{m^3}{sec^2}$	$0.3244 \times 10^{15} \frac{m^3}{sec^2}$	
mass $\frac{Venus}{Sun}$	$\frac{1}{409,650}$	$\frac{1}{408,600}$	(5)
$GM_{Jupiter}$	$1.26464 \times 10^{17} \frac{m^3}{sec^2}$	$1.265 \times 10^{17} \frac{m^3}{sec^2}$	
mass $\frac{Jupiter}{Sun}$	$\frac{1}{1047.355}$	$\frac{1}{1047.355}$	(6)
mass $\frac{Earth\ Moon}{Sun}$	$\frac{1}{328,452}$	$\frac{1}{328,441}$	
J	$1.63808\ 10^{-3}$	1.6246×10^{-3}	(7)
D	$1.0547\ 10^{-5}$	0.6×10^{-5}	(7)
1 A. U.	$1.49503036332 \times 10^{11}$	1.4953×10^{11} m	
		Using $\pi_{\odot} = 8.7984''$	(2)
		and $R_0 = 6378.26$ km	(1)
R_0 (Earth's Equatorial Radius)	6378.206*	6378.260 km (1 and 4)	

*Used with station altitudes, not used in evaluating trajectory constants. (Clarke's spheroid of 1866 is presently used by the United States Coast and Geodetic Survey, Mexico and Canada. Europe, in general, uses Bessel's spheroid of 1841. Therefore, the actual survey of the station will determine which set of dimensions are to be used in the final transformation.)

Table 1 (Cont'd)

Constant	Now Used in JPL Trajectory Program	Recommended Constants	Ref.
1 meter	$\frac{1}{0.3048}$ ft	International Agreement - all countries	
Sidereal Earth Rotation Rate	$0.417807462 \times 10^{-2}$ deg/sec $(0.7292124 \times 10^{-4})$ rad/sec	same	
π_O	8.7934"	8.7934"	(4)
c (speed of light)	299,800 $\frac{km}{sec}$	299,793.0 $\frac{km}{sec}$ ± 1.0	(8)
Moon Radius, Mean	1738.0 km	1738.0 km	(5)
(Radius directed toward earth) - (Polar radius) = a-c	1.08	1.08 km	(5)
(Radius directed towards earth) - (radius in direction of orbit) = a-b	0.2	0.2 km	(5)
J_{Moon}	0.000338	0.000338	(5)
K_{Moon}	0.000035	0.000035	(5)

References Used in Table 1

1. Transactions of the American Geophysical Union, 37:534, 1956.

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3. Herrick, Samuel, Baker, Robert M. L., Jr., and Hilton, Charles G., "Gravitational and Related Constants for Accurate Space Navigation", UCLA Astronomical Papers, I (24): 297-338, 1959.

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5. Allen, C. W., Astrophysical Quantities, University of London, The Athlone Press, 1955.

6. American Ephemeris, 1960 (p. 504), United States Government Printing Office, Washington, D. C.

7. D. C. King-Hele, R. H. Merson, "A New Value for the Earth's Flattening, Derived from Measurements of Satellite Orbits," Nature 183 (4665): 882, March 28, 1959.

8. Cohen, Crowe, Dumond, Fundamental Constants of Physics Interscience Publishers, Incorporated, 1957.

E. Homing Programs

All of the homing programs are based on the use of missile parameters to satisfy the target conditions. Any useful combination of these parameters is generally available as a search option. Further, all of the procedures depend upon the solution of linearized equations in which the coefficients are developed by singly varying the search parameters. In general, if important nonlinearities are suspected, then the option is available to monitor the size of the increments to the search parameters so that they do not exceed values for which the method will converge. The further option of saving these coefficients for successive iterations is also available.

The homing procedure with the integrating program is based on specifying the target conditions in terms of two components of the impact parameter and closest approach time which are computed from the osculating conic about the target mass. One distance component lies in the equatorial plane and is perpendicular to the incoming asymptote to the local hyperbola, whereas, the other component is resolved along the direction mutually perpendicular to the incoming asymptote and the first component. The advantage of using this method of specifying the miss is that these components are relatively linear with respect to variations in the initial conditions.

A subroutine exists which computes the partial derivatives of the impact parameters with respect to the velocity coordinates at prescribed points along the trajectory. As the direction of the velocity perturbation is pertinent, these partials are computed for each of seven directions, including that of each coordinate axis, of the probe-earth line, and of the probe-sun line. The so-called critical directions, in which the miss distance is most sensitive and least sensitive to velocity changes are also determined. In addition, the program computes the velocity increments required to correct the miss.

A homing procedure is available for connecting the burning portion of the trajectory to the interplanetary transfer ellipse program. This is accomplished by matching the right ascension, declination, magnitude of the heliocentric injection velocity vector, and the time at heliocentric injection to the geocentric hyperbola generated by the burning program.

By adjusting the missile parameter selected for variation, the properties of the geocentric hyperbola are changed at the match-point. The matching can also be accomplished by using missile parameters in combination with variations in the heliocentric angle of the transfer ellipse.

F. Tracking

The reduction of tracking data is essentially the problem of filtering, by statistical analysis, the random observational errors and the systematic bias errors.

The basic procedure is as follows: A set of initial conditions is assumed or obtained from iterating within the program and is used to start the integration of the drag-free equations of motion, including the effects of the oblate earth, the moon, and the sun. The computed trajectory variables are transformed into station-referenced coordinates and corrected for refraction and station anomalies. The difference between computed and observed values is used to determine those corrections in initial conditions which result in the minimum sum of squares of the differences between calculations and observations. The corrections in initial conditions are added to the previously employed initial conditions and this completes one iteration. Differential coefficients are computed by integrating a system of equations in which the partial derivatives are the dependent variables.

Prior to full acceptance of a data point into the tracking program, the difference between the computed and observed values is compared with a standard deviation, which is either an externally specified number or one computed within the tracking program from earlier observation points. Measured values which differ from the computed values by more than three times the standard deviation are rejected.

Individual data points are weighted inversely as the variance of the deterioration in quality of the tracking data. The weighting used may depend on whether an automatic tracking mode is used, on the signal strength, and on the elevation angle.

The differences between computations and observations, properly weighted, and the differential coefficients of the observations with respect to the initial conditions, are fed into a number of least-squares-fitting routines. In the primary method, each data type is weighted inversely as the previously computed variances from the mean for that type and then is combined. Changes in the six initial conditions and in constant biases in the five possible observation types can be solved for. Thus, the maximum matrix size provided for is 11 x 11. The matrix size test which can, on option, presently be used is based on the ratio of the changes in initial conditions and biases called for and the computed standard deviation in initial conditions. The new initial conditions and biases obtained by adding the changes solved for are used as input for subsequent trajectory computations. Standard deviations from the mean and standard deviations in initial conditions are always displayed. Standard deviations of predictions are computed on option.

Least-squares routines are applied to each separate data type in order to obtain the changes in initial conditions called for by the various types. The initial conditions so solved for are used to obtain standard deviation from the mean for the optimum fit to each data type separately, and are called "noise" standard deviations. The separate changes in initial conditions obtained above are combined by weighting the results from each type inversely as the variance of that data type from the pointing trajectory. Standard deviations of the initial conditions obtained in this manner are also computed.

G. Ephemeris

The JPL program allows the possibility of introducing ephemerides for the Moon, the Sun, and all planets other than the Earth. Currently, in addition to the Sun and Moon, only Venus, Mars, and Jupiter are being used. All five are available at 1-day intervals relative to the mean equator and equinox of 1950.0.

Present sources for Mercury and Mars are British planetary tables, which do not contain as many significant figures as desired. Sources for the other planets as well as for the Sun are the Themis tapes, which correspond to the best available information from the Naval Observatory. The Moon's tables are relatively difficult to obtain in the 1950.0 coordinate system, and those currently available are not of the desired accuracy. Effort is being made to remedy the deficiencies in the Mars and Moon data.

III. ACCURACY

Sources of inaccuracies must be examined and continually reviewed with respect to changes in targets of interest and requirements for accuracy.

A. Equations of Motion

The equations of motion used can be considered to fall into two categories; those for which analytical solutions exist and those which must be solved by numerical integration. If a problem requires the consideration of more than the simple two-body solution and must be solved by numerical integration, a large increase in complexity and computing time is incurred, which is, to a certain extent, independent of the number of terms used to represent accelerations. In integrating programs, all sources of acceleration which could conceivably be of importance are, therefore, in general included, and no significant errors are incurred because of limitations in the equation integrated.

Conic sections represent solutions to approximate equations of motion, and their use must be commensurate with the expected inaccuracies. Estimates of these inaccuracies can be made by comparing with appropriate numerical integration of more exact equations of motion.

Complete pad-to-target trajectories were generated using the general purpose powered flight program coupled to the heliocentric transfer ellipse program. These trajectories were subsequently computed using an all-integration method. An adjustment of lift-off time of about one hour reduced the miss distance from 5-million km to 500,000 km, which shows the conic program to be weak in keeping track of time.

B. Numerical Integration

In order to derive error estimates for methods of numerical integration, it is necessary that a "standard" or accurate reference solution be available. This standard solution may be either an observed solution, an analytical solution of the same problem, or a numerical solution generated by a method of known accuracy.

Two of the trajectory computation programs were tested by using them to integrate the equations of motion with initial conditions specified by the position and velocity of Mars. Using 16 points clustered around the entries for August 14, 1960, a tenth-order least-squares fit was made to each of the Martian heliocentric coordinates at that time to obtain an initial velocity[*]. This set of initial conditions was integrated using the Milne method and the Gauss-Jackson method of the Themis code. Both programs gave almost identical answers with the error growing in amplitude with time and with a period of approximately one Martian year. It should be noted that within the first six months of elapsed time the maximum difference between ephemeris listings and calculated values was 10,000 km. It appears that the initial conditions derived in the manner described were sufficiently inaccurate to be the principal source of error and to mask all other effects.

In order to use analytic solutions for standards of reference, and as analytic solutions of the relevant n-body problems do not exist, it is necessary to replace the classes of problems of interest by simpler problems which can be solved analytically, with the intention of extrapolating from the error estimates obtained for the simple problems back to the cases of real interest. The extrapolated estimates are reasonable only if it may be assumed that the effect of small

[*]Planetary Co-ordinates for the Years 1960-1980, H. M. Nautical Almanac Office, London, 1958.

perturbing terms in the differential equations on the error in a numerical solution is of the second order and, therefore, may be neglected. Recent investigations at the Laboratory show that this assumption is, in fact, valid for the cases of interest. Error estimates were obtained by comparing the analytical and numerical solutions of the simple two-body problems.

The three numerical methods used in the integration programs discussed were tested for the two-body problem with values of V = 1 (circular orbit), 17/16, 9/8, 5/4, and 11/8, the last value corresponding to an elliptic orbit of eccentricity about 0.89. For each method, the integration step size was varied over a small range from certain nomial values. The integration span was taken to be $0 \leq t \leq 100$, corresponding to about 15 passes for the circular orbit and to about half a period for V = 11/8.

Still another method for estimating errors may be used whenever it is known that the error in the numerical solution may be made "arbitrarily" small by choosing a sufficiently small step size; here a set of numerical solutions of the same problem, but for different step sizes, is obtained and an extrapolation to zero technique is used to obtain a "corrected" solution.

The Runge-Kutta and Milne methods were applied to calculating lunar trajectories with oblateness, Moon's and Sun's perturbing forces included, and extrapolation to zero used to estimate the errors at certain points along the trajectory. These error estimates serve as reasonable estimates for the general case of planetary probes.

The following general conclusions may be drawn:

1. For planetary probes, any of the three methods may be used to provide relative accuracies to orders as low as 10^{-7}. Computing times for a given relative accuracy stand in the ratios 1:8:12 for the Gauss-Jackson, Milne, and Runge-Kutta methods, respectively. Errors build up rapidly, however, if too large a step size is attempted. If only rough estimates to a trajectory are desired, it is better to use approximate methods, such as approximation by segments of conics, rather than numerical integration.
2. None of the methods are suitable for calculating the position of an artificial satellite in its orbit over an appreciable number of periods. Here general perturbation methods (such as Hansen's method) are indicated.
3. On the other hand, experimentation indicates that, for satellite orbits, both the Milne and the Gauss-Jackson methods yield numerical solutions which describe the geometry of the orbit much more accurately than they describe the dynamics; that is, the components x and y lie very close to the orbit, but at the wrong time. Thus, it may be feasible to use numerical integration to study purely geometric properties of satellite orbits; investigation of this phenomenon will continue. The same statement is true of the Runge-Kutta method, but to much less accuracy.

4. In view of the flexibility of the Runge-Kutta method and the fact that, for strict accuracy requirements, the other methods lose some of their time advantage over the Runge-Kutta, use of this method is indicated for precise orbit calculations.
5. Error curves for lunar trajectory tests and for two-body exhibit essentially similar properties, supporting the argument that perturbing forces do not materially affect the numerical error.
6. Error growth seems to be no worse than a linear function of the number of steps taken, even in cases of high error. The best available analytically obtained upper bound on error growth predicts variation proportional to the three-halves power of the number of steps.

C. Ephemeris

A number of problems arise as a result of the fact that numerical integrations are made relative to the mean equator and equinox of 1950.0, whereas, initial conditions and most predicted values of observables are earth-referenced at a specific time. The choice of the coordinate system used for numerical integration is dictated by the requirement that Newton's laws hold only in an inertial frame, and transformation of the equation of motion to the earth-referenced framework in which observations are actually made is virtually untenable. The irregularity of the rotation of the Earth on its axis, as well as the irregularity of the revolution of the Earth about the Sun, give rise to difficulties in transforming back and forth between the two reference systems.

Additional difficulties occur because of the lack of availability from the Naval Observatory of rectangular coordinates of the Moon and of Mars in the 1950.0 rectangular coordinate system. Mars tables are available, but not to as many significant figures as desired. Tables of the Moon explicitly in the rectangular framework are not available.

Still further difficulties exist in accounting for aberrational effects as well as refraction and for various local station anomalies, in regard to making predictions of the values of the observables.

A number of uncertainties arise simply as a result of not knowing physical constants to a high accuracy. For example, studies have shown that errors of the order of one part in 10^4 and one part in 10^5 in the fundamental astronomical unit give rise to errors of 105,000 and 9,900 km, respectively, in the miss distance for a Mars trajectory.

IV. TRAJECTORY DESIGN PROCEDURE

Procedures for designing certain lunar and planetary trajectories have been developed in which the previously described programs are used either directly or in order to generate requirements on the trajectory. A common feature of these procedures is that they attempt to take into account all of the pertinent characteristics of the mission and vehicle system. Therefore, they are flexible and vary considerably in their detailed application. As essentially new trajectory problems arise, modification and extension to existing programs and the initiation of new programs are made. The result is an accurate quantitative description of mission and vehicle system design alternatives in as far as they are trajectory dependent and, finally, the development of a family of standard trajectories.

The procedure will be described by an example in which a specified three-stage vehicle, with parking-orbit capability between the second and third stage, is launched from the Atlantic Missile Range (AMR). The mission is a Mars near-miss to be launched in the latter part of 1960. The communication distance at target arrival is restricted. An initial trajectory carrying near-maximum payload is desired.

A. Preliminary Powered Flight Shaping

The first consideration in the procedure is to determine the conditions at the end of the booster phase of powered flight. This is done in close co-operation with the vehicle designer in order to ensure that aerodynamic forces and the numerous dynamic restrictions of this phase are adequately taken into account. Three or four trajectories having different pitch-program parameters and the same azimuth and launch site are generated. With these booster burnout conditions, the azimuth rotation and launch-site displacement program is used to generate booster burnout conditions of regular firing azimuth intervals for the AMR launch site.

The next step is to determine a pitch program for the upper stages which will give acceptable performance while minimizing vehicle complexity. Using booster burnout conditions at a particular azimuth, the burning program is used to place the third stage in a parking orbit at the end of second-stage burning. The parking orbit is circular at a minimum altitude as analysis shows that this will generally yield a maximum payload. The third stage is burned to a velocity typical for a Mars mission. The thrust optimization program is now used to evaluate the degradation in performance which results from use of the assumed program. By repeating the comparison with the performance of more complicated pitch programs, the trade-off between complexity and performance is established and a particular pitch program is decided upon.

Using the chosen pitch program, position, Earth-fixed velocity, time from lift-off, and weight at second-stage burnout are established as a function of azimuth for entrance into a circular parking orbit. These conditions are stored for later use with the homing programs.

At this point, four missile parameters remain for use in the homing programs: the lift-off time, the firing azimuth, the coast time between second-stage burnout and third-stage ignition (i.e., the time spent in the parking orbit), and the burning time of the last stage.

B. Preliminary Heliocentric Transfer Ellipse

The results of the heliocentric transfer ellipse program are now used. The restriction on communication distance limits the class of trajectories that may be considered by placing an upper limit on the arrival date. In the region considered, the magnitude of the heliocentric injection velocity is still decreasing with respect to increasing arrival time; therefore, for maximum payload, a nominal latest possible arrival date is chosen.

No restrictions have been stated which would limit the heliocentric central angle, so that this parameter is now chosen to minimize the magnitude of the heliocentric injection velocity.

With the arrival date and heliocentric central angle specified, the heliocentric injection time and velocity are specified. By assuming the perigee altitude of the geocentric hyperbola to be approximately the altitude of the circular parking orbit, the launch day and an initial burning time for the last stage are determined.

The initial firing azimuth can now be chosen by setting the inclination of the parking orbit equal to the declination of the heliocentric injection velocity vector. It is clear that solutions will occur for inclinations greater than this, but with maximum payload as the prime consideration the firing azimuth should be as easterly as possible.

The initial value for the coast time is now determined by igniting the last stage at a latitude which is the negative of the declination of the heliocentric velocity vector. This will give the declination of the asymptote to the geocentric hyperbola approximately the same value as that of the heliocentric injection velocity vector.

C. Conic Section Homing

All of the initial conditions for entering the heliocentric transfer ellipse homing program are now available. Since it is already clear where the firing azimuth for maximum payload will occur, this missile parameter is not used in the search. The option involving launch time, coast time between second and third stage, burning time of the last stage, and the heliocentric central angle of the transfer ellipse is used.

Upon converging, one additional case is constructed with the initial heliocentric angle decreased by 1 deg. This will make the launch day one day later. The inclination required on the parking orbit is somewhat smaller so that the firing azimuth is now more easterly, although the heliocentric injection velocity is higher. Thus, the azimuth change tends to increase the payload, whereas, the increased energy in orbit tends to decrease it. By successive cases of this type, the maximum payload conditions are ascertained.

D. Integrated Trajectory

The converged missile parameters giving the maximum payload condition in the conic section homing program are now used as initial conditions for the integrating programs.

By previous experience, it is known that the major contribution to the target miss distance will be in the lift-off time, therefore, the initial search option chosen will assign the right ascension component of the miss to lift-off time. The correction on lift-off time will be made once, as this proves to be most economical of search time in most cases.

The second step in the homing procedure involves searching on the three parameters, lift-off time, coast time, and burning time of the last stage, to achieve the final integrated hitting trajectory for a given communication distance.